Bending and Extension of a Multilayered Plate with Body Forces and Thermal Loading

Hyeon Gyu Beom* and Ki Ju Kang*

(Received December 28, 1998)

A multilayered plate composed of thin layers of isotropic materials is analyzed. The problem for the multilayered plate with body forces is formulated by using the lamination theory in which displacement fields are expressed in terms of in-plane displacements on a main plane and transverse displacement. Placing the main plane at an appropriate distance from the lower surface of the plate, a set of equilibrium equations is shown to be written in uncoupled forms, which are identical to those for an uncoupled plate such as a single layer plate. It is proved that the complete solutions of the multilayered plates subject to the specified in-plane resultant tractions or in-plane displacements on its whole boundary can be obtained from the sum of solutions for uncoupled plates. Closed form solutions are obtained for a circular laminate clamped or simply supported on its the boundary as well as for a rotating disk with a constant angular velocity. The calculations of thermoelastic stresses and displacements in multilayered plates are also discussed. Closed form solutions are obtained for a circular laminate with distributed temperature varying in the radial direction and through the thickness.

Key Words : Multilayered Plate, Bending Thermoelastic Stress, Body Forces, Thermal Loading, Circular Laminate

1. Introduction

Thin multilayers bonded together and compositionally graded thin films have found a variety of engineering applications in the fields of microelectronics, optoelectronics and magnetic storage devices (Nix, 1989; Geiger, 1992; Koizumi, 1992). Deformation of a plate composed of thin layers has recently gained importance, due to its potential application in various kinds of technological problems (Kang and Cho, 1991; Finot and Suresh, 1996; Freund, 1996; Kim and Yoon, 1997). Studies on warpage of thin plastic electronic packages have been performed by Suhir and Manzione (1993) and Suhir (1993). Thermal stress in a multilayered strip is an important problem in electronic devices (Lau, 1993; Tummala et al., 1997). Thermoelastic responses of compositionally graded multilayers under small deformation conditions have been addressed theoretically (Nakajima, 1992; Freund, 1996). These works, however, are concerned with a plate with a distributed temperature varying only through the thickness.

The analysis of an uncoupled plate with body forces and thermal loading such as a single layer plate and a symmetric laminate usually entails two uncoupled problems, e.g., the one of pure in-plane deformations and the other of transverse deformations. However, in-plane and transverse displacements of a laminated plate are, in general coupled. This coupling makes it difficult to obtain the exact closed form solutions, while analytical solutions for a single layer can be generated without too much difficulty in some instances because of simpler mathematics involved. Many solutions of uncoupled plates can be found in Young (1989), in which the possible effect of coupling between the in-plane displacement and the transverse displacement has not yet

* Department of Mechanical Engineering, College of Engineering, Chonnam National University, 300, Yongbong-dong, Kwangju, 500-757, Korea.

been dealt with. Few problems of laminated plates composed of multiple isotropic layers have been solved analytically since those require solving the coupled equations, making the task quite complicated.

It is the purpose of this study to investigate the problem of a plate composed of thin layers of isotropic materials. The problem for the multilayered plate with body forces is formulated by using plate theory in which displacement fields are expressed in terms of in-plane displacements on a main plane and transverse displacement. In the classical lamination theory, displacement fields in a laminated plate can be expressed in terms of in-plane displacements on the middle surface and transverse displacement. The equilibrium equations in the classical lamination theory are coupled with respect to the in-plane displacements on the middle surface and the transverse displacement. This paper is devoted to derivation of laminated plate theory in which the equilibrium equations are uncoupled, and are identical to those for an uncoupled plate such as a single layer plate. The boundary conditions, in contrast to an uncoupled plate, depend on the in-plane displacement as well as on the transverse displacement. Two kinds of problems for the multilayered plate with body forces, which are of particular importance in the practical application, are considered in this paper: The first one is a laminate subjected to the in-plane resultant tractions prescribed on its boundary. The second one is a laminate subjected to the specified in-plane displacements on its boundary. It will be proved that the complete solutions for the problems of the multilayered plates can be obtained from the sum of solutions for uncoupled plates. As examples a circular laminate clamped or simply supported on its the boundary as well as a rotating disk with a constant angular velocity is considered. Closed form solutions are obtained from the known solution of an uncoupled plate by applying the superposition method proposed in this paper. The calculations of thermoelastic stresses and displacements in multilayered plates are also discussed. Closed form solutions are obtained for a circular laminate with distributed temperature varying in the radial direction and through the thickness.

2. Formulation of a Multilayered Plate

Consider a deformation of a plate composed of thin layers of isotropic material. The cross section of the plate is shown schematically in Fig. 1. The plate has a constant thickness h. According to the classical lamination theory, the displacements at any point of a plate are written as

$$u_i = u_i^0 - x_3 w_{,i}(x_1, x_2), \quad (i=1,2)$$
$$u_3 = w(x_1, x_2) \tag{1}$$

where $u_i (i=1, 2)$ and u_3 are the in-plane and transverse displacements, respectively; u_i^0 is the in-plane displacement on the main plane which will be defined exactly later, and the subscript comma(,) denotes a partial derivative with respective to the in-plane Cartesian coordinates, x_1 and x_2. In the classical lamination theory (Jones, 1975), displacement fields in a laminated plate can be expressed in terms of in-plane displacements on the middle surface and transverse displacement. The equilibrium equations in the lamination theory are coupled with respect to the in-plane displacements on the middle surface and the transverse displacement. However, the lamination theory derived in this paper leads to a set of uncoupled equilibrium equations, which are identical to those for an uncoupled plate.

The Hooke's law relating the stresses σ_{ij} and strains ε_{ij} for a layer composing the plate can be written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

Here C_{ijkl} is the stiffness tensor given as

$$C_{ijkl} = C_{12}\delta_{ij}\delta_{kl} + \frac{1}{2}(C_{11} - C_{12})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{3}$$

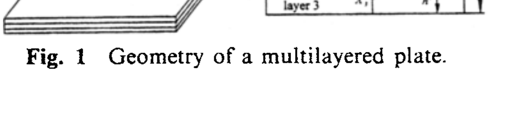


Fig. 1 Geometry of a multilayered plate.

where δ_{ij} is the Kronecker delta, and C_{11} and C_{12} and can be written in terms of Young's modulus E and Poisson's ratio v of the layer as $C_{11}=\frac{E}{1-v^2}$ and $C_{12}=\frac{vE}{1-v^2}$. In this paper, the repetition of an index in a term denotes a summation with respect to that index over its range 1 to 2 for a Roman letter lowercase, unless indicated otherwise. Without loss of generality, we can place the main plane ($x_3=0$) at the distance

$$h^0=\frac{\int_0^h X_3 C_{11} dX_3}{\int_0^h C_{11} dX_3} \tag{4}$$

from the lowest surface of the plate. Here, $X_3(=x_3+h^0)$ is the vertical coordinate of the given point from the lowest surface of the plate. Integrating the stresses given in (2) through the thickness we have the following constitutive relations for the laminated plate;

$$\begin{aligned} N_{ij}&=A_{ijkl}\varepsilon^0_{kl}+B_{ijkl}\chi_{kl} \\ M_{ij}&=B_{ijkl}\varepsilon^0_{kl}+D_{ijkl}\chi_{kl} \end{aligned} \tag{5}$$

Here N_{ij} and M_{ij} are the resultant force and moment defined as $N_{ij}=\int_{-h^0}^{h-h^0}\sigma_{ij}dx_3$ and $M_{ij}=\int_{-h^0}^{h-h^0}x_3\sigma_{ij}dx_3$, respectively. ε^0_{kl} and χ_{kl} are the main plane strain tensor and the curvature defined as $\varepsilon^0_{kl}=\frac{1}{2}(u^0_{k,l}+u^0_{l,k})$ and $\chi_{kl}=-w_{,kl}$, respectively. A_{ijkl}, B_{ijkl} and D_{ijkl} are the extensional, coupling and bending stiffness tensors, respectively, given as

$$\begin{aligned} A_{ijkl}&=A_{12}\delta_{ij}\delta_{kl}+\frac{1}{2}(A_{11}-A_{12})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \\ B_{ijkl}&=B_{12}\delta_{ij}\delta_{kl}-\frac{1}{2}B_{12}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \\ D_{ijkl}&=D_{12}\delta_{ij}\delta_{kl}+\frac{1}{2}(D_{11}-D_{12})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \end{aligned} \tag{6}$$

where

$$\begin{aligned} A_{ij}&=\int_{-h^0}^{h-h^0}C_{ij}dx_3, \\ B_{ij}&=\int_{-h^0}^{h-h^0}x_3C_{ij}dx_3, \text{ and} \\ D_{ij}&=\int_{-h^0}^{h-h^0}x_3^2C_{ij}dx_3, \quad (ij=11,\ 12) \end{aligned} \tag{7}$$

It is noted that the formulation based on the through-thickness averaging approach in the lamination theory is also applicable to a problem for a graded material. Clearly, Eq. (4) is equivalent to the condition

$$B_{11}=\int_{-h^0}^{h-h^0}x_3C_{11}dx_3=0 \tag{8}$$

The presence of the coupling stiffness, B_{ijkl} implies the coupling between the bending and extension of a laminated plate.

The equations of equilibrium for the laminated plate are

$$\begin{aligned} N_{ij,j}+p_i&=0 \\ Q_{i,i}+p_3&=0 \end{aligned} \tag{9}$$

Here p_i is the resultant in-plane body force given by $p_i=\int_{-h^0}^{h-h^0}f_idx_3$, in which f_i is the in-plane body force per unit volume and p_3 is the transverse force acting on the plate. Q_i is the transversal shearing force, which can be obtained from

$$Q_i=M_{ij,j}+m_i \tag{10}$$

where $m_i=\int_{-h^0}^{h-h^0}x_3f_idx_3$. Substituting (5) into (9), a set of equilibrium equations can be written in uncoupled form

$$\begin{aligned} &\frac{1}{2}(A_{11}+A_{12})u^0_{j,ji}+\frac{1}{2}(A_{11}-A_{12})u^0_{i,jj}+p_i=0 \\ &D_{11}w_{,iijj}-m_{i,i}-p_3=0 \end{aligned} \tag{11}$$

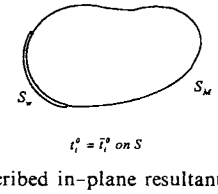
It is interesting to note that the equilibrium equations for a coupled plate are identical to those for an uncoupled plate such as a single layer plate or a symmetric laminate. For a unique solution of Eq. (11), it is sufficient to specify the following boundary conditions along the plate boundary S

$$\begin{aligned} &u^0_i=\bar{u}^0_i \text{ on } S_u \\ &t^0_i=\bar{t}^0_i \text{ on } S_t \\ &w=\bar{w},\ w_{,n}=\bar{w}_{,n} \text{ on } S_w \\ &V_n=\bar{V}_n,\ M_{nn}=\bar{M}_{nn} \text{ on } S_M \text{ (not summed on } n) \end{aligned} \tag{12}$$

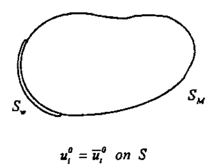
Here t^0_i is the resultant in-plane traction given as $t^0_i=N_{ij}n_j$ and V_n is the Kirchhoff force at an edge for a plate defined as $V_n=Q_n+\frac{\partial M_{ns}}{\partial s}$, where n and s refer to the directions normal and tangential to the plate edge, respectively. An overscript bar ($\bar{}$) designates the prescribed value and $S=S_u\cup S_t=S_w\cup S_M$. Details for derivation of the boundary conditions are presented in Jones (1975) and Timoshenko and Woinowsky-Krieger (1959). Note, in contrast to a single layer, that the boundary conditions involve the in-plane displacement as well as on the transverse displacement. The only difference in appearance between a single layer and a laminated layer is that the latter has the coupling between the bending and extension in the constitutive equation and boundary conditions whereas the single layer has no coupling.

3. Multilayered Plate with Body Forces

Two kinds of problems for the multilayered and graded materials with body forces as shown in Fig. 2, which are of particular importance in the practical application, are considered in this section. The first one is a laminate subjected to the in-plane resultant tractions prescribed on its boundary. The second one is a laminate subjected to the specified in-plane displacements on its boundary. It will be proved that the complete solutions of displacements for the problems can be obtained by superposition of known solutions for an uncoupled plate, due to the linearity of the problem. Once solutions of the displacements are obtained, the corresponding resultant stresses and moments can be evaluated from Eq. (5).



(a) Prescribed in-plane resultant tractions.



(b) Prescribed in-plane displacements.

Fig. 2 Multilayered plate with body forces.

3.1 Laminate subjected to in-plane resultant tractions

Consider a problem of a simply connected plate with in-plane resultant tractions prescribed on its boundary. The boundary conditions can be written as

$$\begin{aligned} &t_i^0=\bar{t}_i^0 \text{ on } S \\ &w=\bar{w},\quad w_{,n}=\bar{w}_{,n} \text{ on } S_w \\ &V_n=\bar{V}_n,\quad M_{nn}=\bar{M}_{nn} \text{ on } S_M \end{aligned} \tag{13}$$

The boundary conditions (13) reduce to various commonly encountered special cases such as a simply supported plate when the prescribed quantities are taken to be zero. Applying the superposition principle, we can obtain solutions of the plate from the sum of solutions for uncoupled plates. The superposition applied to a laminated plate is illustrated in Fig. 3. The displacements for the plate are written as

$$\begin{aligned} &u_i^0=u_i^{0(a)}+u_i^{0(b)}+u_i^{0(c)} \\ &w=w^{(a)}+w^{(b)}+w^{(c)} \end{aligned} \tag{14}$$

where superscripts a, b and c in parentheses indicate the quantities associated with the problems a, b and c, respectively. For the problem a, $p_3^{(a)}=p_3$, $m_i^{(a)}=m_i$ and $u_i^{0(a)}\equiv 0$. The boundary conditions for the problem a are given as

$$\begin{aligned} &w^{(a)}=\bar{w},\ w_{,n}^{(a)}=\bar{w}_{,n} \text{ on } S_w \\ &V_n^{(a)}=\bar{V}_n,\ M_{nn}^{(a)}=\bar{M}_{nn} \text{ on } S_M \end{aligned} \tag{15}$$

The solution of the transverse displacement to (11) and (15) is given by $w^{(a)}=\hat{w}^{(a)}$, where overscript hat($\hat{}$) represents the quantities associated with the problem of uncoupled plate with $B_{ijkl}=0$ and same boundary conditions. For the problem b, $p_i^{(b)}=p_i$ and $w^{(b)}\equiv 0$. The boundary conditions for the problem b are

$$t_i^{0(b)}=\bar{t}_i^0-t_i^{0(a)} \text{ on } S \tag{16}$$

The solution of the in-plane displacement satisfying (11) and (16) is given by $u_i^{0(b)}=\hat{u}_i^{0(b)}$ where $\hat{u}_i^{0(b)}$ is the solution for the uncoupled plate subjected to the boundary condition (16). The boundary conditions appropriate to the problem c is given by

$$\begin{aligned} &t_i^{0(c)}=0 \text{ on } S \\ &w^{(c)}=0,\ w_{,n}^{(c)}=0 \text{ on } S_w \\ &V_n^{(c)}=-V_n^{(b)},\ M_{nn}^{(c)}=-M_{nn}^{(b)} \text{ on } S_M \end{aligned} \tag{17}$$

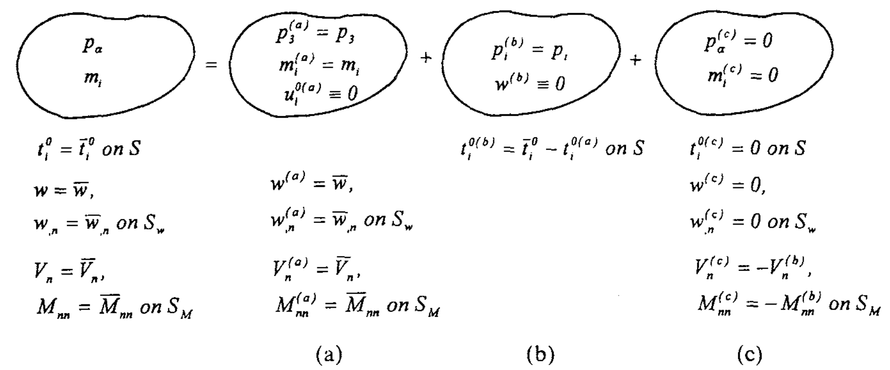


Fig. 3 Application of superposition to obtain solutions of a laminate subjected to in-plane resultant tractions.

In addition, $p_\alpha^{(c)}=0(\alpha=1, 2, 3)$ and $m_i^{(c)}=0$. The in-plane problem for the plate subjected to the boundary condition (17) can be solved by taking $N_{ij}^{(c)}=0$ throughout the plate, which satisfies the in-plane boundary condition (17) as well as the equilibrium Eq. (9). Inversion of (5) for $N_{ij}^{(c)}=0$ gives the main plane strains

$$\varepsilon_{ij}^{0(c)}=\frac{B_{12}}{A_{11}^2-A_{12}^2}[A_{11}w_{,kk}^{(c)}\delta_{ij}-(A_{11}+A_{12})\,w_{,ij}^{(c)}] \tag{18}$$

The main plane strains satisfy a compatibility requirement, which in the case of a simply connected plate reduce to the single relation

$$\varepsilon_{ii,jj}^{0(c)}-\varepsilon_{ij,ij}^{0(c)}=0 \tag{19}$$

In obtaining (19), it has been assumed implicitly that the transverse displacement $w^{(c)}$ satisfies the equilibrium equation, $w_{,iijj}^{(c)}=0$. Substitution of (18) into (5) yields

$$M_{ij}^{(c)}=D_{ijkl}^{*}\chi_{kl}^{(c)} \tag{20}$$

where the bending stiffness D_{ijkl}^{*} is defined as

$$D_{ijkl}^{*}=D_{12}^{*}\delta_{ij}\delta_{kl}+\frac{1}{2}(D_{11}^{*}-D_{12}^{*})\,(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

$$D_{11}^{*}=D_{11}-\frac{A_{11}B_{12}^2}{A_{11}^2-A_{12}^2}$$

$$D_{12}^{*}=D_{12}+\frac{A_{12}B_{12}^2}{A_{11}^2-A_{12}^2} \tag{21}$$

Thus, the transverse displacement for the problem c is written as

$$w^{(c)}=\hat{w}^{*} \tag{22}$$

where $\hat{w}^{*}$ is the solution for a uncoupled plate with the bending stiffness D_{ijkl}^{*}, subject to the boundary conditions (17). The main plane displacements can be obtained from (18). Recently, the closed form solution of the in-plane displacements for the problem c have been solved by using the complex potential method in Beom and Earmme (1998).

As shown above, the complete solutions of displacements for the problem can be obtained from the sum of solutions for an uncoupled plate. Once solutions of the displacements are obtained, the corresponding resultant stresses and moments can be evaluated from (5).

As an example we consider a simply supported circular laminate with radius a. The plate is subjected to a uniformly distributed transverse load given by $p_3=p_3^0$, as shown in Fig. 4(a). We introduce cylindrical coordinates (r, θ) for convenience, and take the origin of coordinates at the center of the plate and denote by r the radial distances of the points in the main plane of the plate. All field quantities are independent of θ and are function of r alone. Making use of the known solution of the uncoupled plate found in Timoshenko and Woinowsky-Krieger (1959), it can be shown that the solutions of the plate are written as

$$\begin{aligned}
u_r^0&=-\frac{p_3^0a^2Br}{8AD(1+v^A)(1+v^D)}\left[1+\frac{B^2}{AD^*(1+v^A)(1+v^{D^*})}\right]\\
u_\theta^0&=0 \qquad\qquad (23)\\
w&=\frac{p_3^0}{64D}(a^2-r^2)\left[\frac{5+v^D}{1+v^D}a^2\right.\\
&\left.+\frac{4a^2B^2}{AD^*(1+v^D)(1+v^{D*})(1+v^A)}-r^2\right]
\end{aligned}$$

where $A=A_{11}$, $v^A=\dfrac{A_{12}}{A_{11}}$, $B=B_{12}$, $D=D_{11}$,

(a) Uniform transverse loading $p_3=p_3^0$. (b) Rotating disk with angular velocity ω.

Fig. 4 Circular plates.

$v^D=\dfrac{D_{12}}{D_{11}}$ and $r=\sqrt{x_k x_k}$

In obtaining (23), the superposition method for the problem provided in this paper has been used. Substituting (23) into (5) provides the following expressions for the resultant stresses and moments

$$
\begin{aligned}
&N_{rr}=\frac{p_3^0 B}{16D}(a^2-r^2)\\
&N_{\theta\theta}=\frac{p_3^0 B}{16D}(a^2-3r^2)\\
&N_{r\theta}=0\\
&M_{rr}=\frac{1}{16}p_3^0(3+v^D)(a^2-r^2)\\
&M_{\theta\theta}=\frac{1}{16}p_3^0[(3+v^D)a^2-(1+3v^D)r^2]\\
&M_{r\theta}=0
\end{aligned}
\tag{24}
$$

As a second example we consider a disk rotating at an angular velocity ω, as shown in Fig. 4 (b). The body forces per unit volume are $f_r=\rho\omega^2 r$ and $f_\theta=0$ where Le is the density. Thus, the resultant body forces and moment due to the body forces are $p_r=p^\omega r$, $p_\theta=0$, $m_r=m^\omega r$ and $m_\theta=0$, where $p^\omega=\int_{-h^0}^{h-h^0}\rho\omega^2 dx_3$ and $m^\omega=\int_{-h^0}^{h-h^0}\rho\omega^2 x_3 dx_3$. Application of the superposition principle results in

$$
\begin{aligned}
u_r^0=&\frac{p^\omega}{8A}\left[-p^\omega r^3+\frac{3+v^A}{1+v^A}p^\omega a^2 r-\frac{2m^\omega B a^2 r}{D(1+v^D)(1+v^A)}\right]\\
&-\frac{BM^0 r}{AD^*(1+v^A)(1+v^{D*})},\\
u_\theta^0=&0\\
w=&\frac{m^\omega}{32D}\left(r^4-2\frac{3+v^D}{1+v^D}a^2r^2\right)-\frac{M^0 r}{2D^*(1+v^{D*})}
\end{aligned}
\tag{25}
$$

where $M^0=\dfrac{a^2B}{4A}\left[-\dfrac{p^\omega}{1+v^A}+\dfrac{m^\omega B}{D(1+v^D)(1+v^A)}\right]$. The corresponding expressions for the resultant stresses and moments are

$$
\begin{aligned}
&N_{rr}=\frac{1}{8}\left[p^\omega(3+v^A)+\frac{m^\omega B}{D}\right](a^2-r^2)\\
&N_{\theta\theta}=\frac{p^\omega}{8}[(3+v^A)a^2-(3v^A+1)r^2]\\
&\qquad+\frac{m^\omega B}{8D}(a^2-3r^2)\\
&N_{r\theta}=0\\
&M_{rr}=\frac{m^\omega}{8}(3+v^D)(a^2-r^2)+\frac{p^\omega B}{8A}(a^2-r^2)\\
&M_{\theta\theta}=\frac{m^\omega}{8}[(3+v^D)a^2-(3v^D+1)r^2]\\
&\qquad+\frac{p^\omega B}{8A}(a^2-3r^2)\\
&M_{r\theta}=0
\end{aligned}
\tag{26}
$$

3.2 Laminate subjected to in-plane displacements

Consider a problem of a simply connected plate with in-plane displacements prescribed on its boundary. The boundary conditions are

$$
\begin{aligned}
&u_i^0=\bar{u}_i^0 \text{ on } S\\
&w=\bar{w},\ w_{,n}=\bar{w}_{,n} \text{ on } S_w\\
&V_n=\bar{V}_n,\ M_{nn}=\bar{M}_{nn} \text{ on } S_M
\end{aligned}
\tag{27}
$$

The boundary conditions (27) reduce to various commonly encountered special cases such as a clamped plate when the prescribed quantities are taken to be zero. Applying the superposition principle, we can obtain solutions of the plate from the sum of solutions for uncoupled plates. The superposition applied to a laminated plate is illustrated in Fig. 5. The displacements for the plate are written as

$$
\begin{aligned}
&u_i^0=u_i^{0(a)}+u_i^{0(b)}\\
&w=w^{(a)}+w^{(b)}
\end{aligned}
\tag{28}
$$

where superscripts a and b in parentheses indicate the quantities associated with the problems a and b, respectively. For the problem a, $p_i^{(a)}=p_i$ and $w^{(a)}\equiv 0$. The boundary conditions for the problem a are

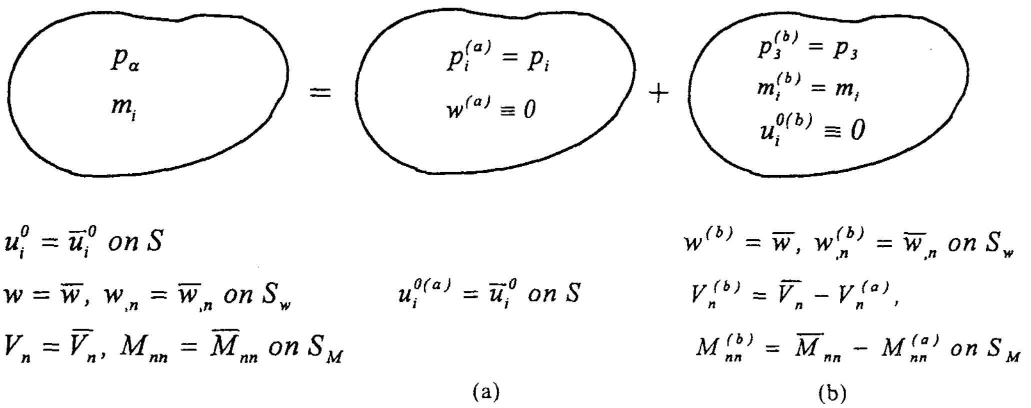


Fig. 5 Application of superposition to obtain solutions of a laminate subjected to in-plane displacements.

$$u_i^{0(a)} = \bar{u}_i^0 \text{ on } S \tag{29}$$

The solution of the in-plane displacement satisfying (11) and (29) is given by $u_i^{0(a)} = \hat{u}_i^{0(a)}$. For the problem b, $p_3^{(b)} = p_3$, $m_i^{(b)} = m_i$ and $u_i^{0(b)} \equiv 0$. The boundary conditions for the problem b are given as

$$\begin{aligned} &w^{(b)} = \bar{w},\ w_{,n}^{(b)} = \bar{w}_{,n} \text{ on } S_w \\ &V_n^{(b)} = \bar{V}_n - V_n^{(a)},\ M_{nn}^{(b)} = \bar{M}_{nn} - M_{nn}^{(a)} \\ &\text{on } S_M \end{aligned} \tag{30}$$

The solution of the transverse displacement to (11) and (30) is given by $w^{(b)} = \hat{w}^{(b)}$.

For a special case of the problem, we now consider a plate subjected to the following boundary conditions

$$\begin{aligned} &u_i^0 = \bar{u}_i^0 \text{ on } S \\ &w = \bar{w},\ w_{,n} = \bar{w}_{,n} \text{ on } S \end{aligned} \tag{31}$$

There is no coupling between in-plane displacement u_i^0 and transverse displacement w in the equilibrium equations and the boundary conditions. Thus, the problem of in-plane stretching and transverse bending of the plate can be treated independently. The displacements u_i^0 and w may be written as

$$\begin{aligned} &u_i^0 = \hat{u}_i^0 \\ &w = \hat{w} \end{aligned} \tag{32}$$

The displacements u_i^0 and w are identical for the plate with $B_{ijkl} = 0$, respectively, while the resultant stresses and moments depend on the coupling stiffness.

We consider a circular plate subjected to a uniformly distributed transverse load $p_3 = p_3^0$. The plate is clamped at its boundary $r = a$. Applying the superposition principle as mentioned above, it can be shown that the solutions of the plate are

$$\begin{aligned} &u_r^0 = u_\theta^0 = 0 \\ &w = \frac{p_3^0}{64D}(a^2 - r^2)^2 \end{aligned} \tag{33}$$

In obtaining (33), the solutions for the corresponding uncoupled plate found in Timoshenko and Woinowsky-Krieger (1959) have been used. Since the displacements are determined as above, we can calculate the associated resultant stresses and moments from (5) and (33). The corresponding resultant stresses and moments are written as

$$\begin{aligned} &N_{rr} = \frac{p_3^0 B}{16D}(a^2 - r^2) \\ &N_{\theta\theta} = \frac{p_3^0 B}{16D}(a^2 - 3r^2) \\ &N_{r\theta} = 0 \\ &M_{rr} = \frac{p_3^0}{16}[(1 + v^D)a^2 - (3 + v^D)r^2] \\ &M_{\theta\theta} = \frac{p_3^0}{16}[(1 + v^D)a^2 - (1 + 3v^D)r^2] \\ &M_{r\theta} = 0 \end{aligned} \tag{34}$$

It is interesting to note that the displacements (33) and the corresponding resultant moment M_{ij} are identical to the solutions of a single layer found in Timoshenko and Woinowsky-Krieger (1959). The resultant stresses corresponding to the displacement fields, however, depend on the coupling stiffness.

4. Thermal Stress in a Multilayered Plate

In the preceding section, the basic formulations of problems of a multilayered plate with body forces were derived. The calculations of thermoelastic stresses and displacements in thin plates discussed in this section bear a close resemblance to the corresponding ones of isothermal plate theory. Employing the Duhamel-Neumann analogy which is in fact an extension to the problem of the laminate, solutions of multilayered plate problems can be reduced to those the corresponding isothermal plates with body forces. Consequently, all analytical method described in the preceding sections are applicable to layered plates with temperature distributions.

We consider again a deformation of a plate composed of thin layers of isotropic linear elastic material. The cross section of the plate is shown schematically in Fig. 1. The plate has constant thickness h. A temperature change T from the initial stress-free state varies in the plate. The Duhamel-Neumann law relating the stresses σ_{ij} and strains ε_{ij} for each layer composing the plate can be written as

$$\sigma_{ij}=C_{ijkl}(\varepsilon_{kl}-\varepsilon^T_{kl}) \tag{35}$$

Here $\varepsilon^T_{ij}=\alpha T\delta_{ij}$ in which α is the coefficient of thermal expansion. Integrating the stresses given in (35) through the thickness, the constitutive relations for the laminated plate have the following form

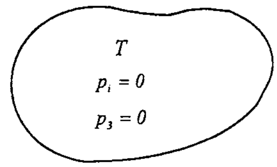
$$\begin{aligned} N_{ij}&=A_{ijkl}\varepsilon^0_{kl}+B_{ijkl}\chi_{kl}-N^T_{ij} \\ M_{ij}&=B_{ijkl}\varepsilon^0_{kl}+D_{ijkl}\chi_{kl}-M^T_{ij} \end{aligned} \tag{36}$$

where

$$\begin{aligned} N^T_{ij}&=\int_{-h^0}^{h-h^0}(C_{11}+C_{12})\,\alpha T dx_3\delta_{ij} \\ M^T_{ij}&=\int_{-h^0}^{h-h^0}(C_{11}+C_{12})\,\alpha T x_3 dx_3\delta_{ij} \end{aligned} \tag{37}$$

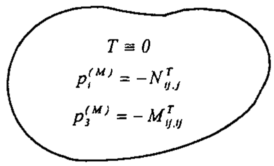
Neglecting body forces and transverse loading, a set of equilibrium equations can be written in uncoupled form

$$\frac{1}{2}(A_{11}+A_{12})\,u^0_{j,ji}+\frac{1}{2}(A_{11}-A_{12})\,u^0_{i,jj}$$



$$u^0_i=\bar{u}^0_i \ on\ S_u,\ t^0_i=\bar{t}^0_i \ on\ S_t$$
$$w=\bar{w},\ w_{,n}=\bar{w}_{,n}\ on\ S_w$$
$$V_n=\bar{V}_n,\ M_{nn}=\bar{M}_{nn}\ on\ S_M$$

(a) Thermoelastic problem



$$u^{0(M)}_i=\bar{u}^0_i \ on\ S_u,\ t^{0(M)}_i=\bar{t}^0_i+t^{0T}_i\ on\ S_M$$
$$w^{(M)}=\bar{w},\ w^{(M)}_{,n}=\bar{w}_{,n}\ on\ S_w$$
$$V^{(M)}_n=\bar{V}_n+V^T_n,\ M^{(M)}_{nn}=\bar{M}_{nn}+M^T_{nn}\ on\ S_M$$

(b) Mechanical problem

Fig. 6 Duhamel-Neumann analogy.

$$\begin{aligned} &-N^T_{ij,j}=0 \\ &D_{11}w_{,iijj}+M^T_{ij,ij}=0 \end{aligned} \tag{38}$$

It is noted that the equilibrium Eq. (38) are identical to those for an uncoupled plate.

Employing the Duhamel-Neumann analogy which is in fact an extension to the problem of the laminate, the displacements produced by a temperature change can be obtained from those produced by body forces. By comparing (38) with the corresponding equations in (11) for the plate with body forces, respectively, we see that the effect of the temperature change T is equivalent to replacing p_i and p_3 by $-N^T_{ij,j}$ and $-M^T_{ij,ij}$, respectively. The Duhamel-Neumann analogy applied to the laminated plate is illustrated in Fig. 6. Thus, we consider a mechanical problem M of a plate of exactly same shape but with the following conditions

$$\begin{aligned} &T^{(M)}\equiv 0, \\ &p^{(M)}_i=-N^T_{ij,j},\quad p^{(M)}_3=-M^T_{ij,ij}, \\ &u^{0(M)}_i=\bar{u}^0_i \text{ on } S_u,\quad t^{0(M)}_i=\bar{t}^0_i+t^{0T}_i \text{ on } S_t, \end{aligned}$$

$$w^{(M)}=\bar{w},\quad w_{,n}^{(M)}=\bar{w}_{,n} \text{ on } S_w,$$
$$V_n^{(M)}=\bar{V}_n+V_n^T,\ M_{nn}^{(M)}=\bar{M}_{nn}+M_{nn}^T \text{ on } S_M \quad (39)$$

where $t_i^{0T}=N_{ij}^T n_j$ and V_n^T is the Kirchhoff force due to the thermal moment M_{ij}^T. Then, the displacement, and the resultant stress and moment for the laminate with thermal loading can be written as

$$\begin{aligned} u_i^0&=u_i^{0(M)}\\ w&=w^{(M)}\\ N_{ij}&=N_{ij}^{(M)}-N_{ij}^T\\ M_{ij}&=M_{ij}^{(M)}-M_{ij}^T \end{aligned} \quad (40)$$

As seen in the previous section, the complete solutions of displacements for the mechanical problem M can be obtained from the sum of solutions for an uncoupled plate. Once solutions of the displacements for the mechanical problem M are obtained, the corresponding resultant stresses and moments can be evaluated from (40).

As an example, we consider a circular plate with radius a. The distributed temperature change T varies only in the r and x_3 directions so that the thermal stress and moment are $N_{ij}^T=N_{ij}^T(r)$ and $M_{ij}^T=M_{ij}^T(r)$. In such a case the main plane displacements and transverse displacement are independent of θ and are function of r alone. When the plate is simply supported at its boundary $r=a$, it can be shown that the solutions of the plate are written as

$$\begin{aligned} u_r^0&=(1+v^A)\,r\phi(r)\\ &\quad+\left[(1-v^A)\,\phi(a)-\frac{2B(1+v^D)}{A(1+v^A)}\psi(a)\right]r\\ &\quad-\frac{BM_0^T r}{AD^*(1+v^A)(1+v^{D*})}\\ u_\theta^0&=0\\ w&=-(1+v^D)\int_0^r \xi\psi(\xi)\,d\xi-\frac{1-v^D}{2}\psi(a)\,r^2\\ &\quad-\frac{M_0^T r^2}{2D^*(1+v^{D*})} \end{aligned} \quad (41)$$

where

$$\begin{aligned} \phi(r)&=\frac{1}{r^2}\int_0^r \xi\varepsilon^{0T}(\xi)\,d\xi,\\ \psi(r)&=\frac{1}{r^2}\int_0^r \xi x^T(\xi)\,d\xi\\ M_0^T&=-2B\phi(a)+\frac{2B^2(1+v^D)}{A(1+v^A)}\psi(a) \end{aligned} \quad (42)$$

$$\varepsilon^{0T}=\frac{1}{A_{11}+A_{12}}\int_{-h^0}^{h-h^0}(C_{11}+C_{12})\,\alpha T dx_3,$$
$$x^T=\frac{1}{D_{11}+D_{12}}\int_{-h^0}^{h-h^0}(C_{11}+C_{12})\,\alpha T x_3 dx_3$$

In obtaining (23), the superposition method for the problem provided in this paper has been used. Substituting (23) into (40) provides the following expressions for the resultant stresses and moments

$$\begin{aligned} N_{rr}&=A(1-v^{A2})[\phi(r)-\phi(a)]+B(1+v^D)\\ &\quad[\psi(r)-\psi(a)]\\ N_{\theta\theta}&=A(1-v^{A2})[\phi(r)+\phi(a)]-B(1+v^D)\\ &\quad[\psi(r)+\psi(a)-x^T]\\ N_{r\theta}&=0\\ M_{rr}&=D(1-v^{D2})[\psi(a)-\psi(r)]-B(1+v^A)\\ &\quad[\phi(a)-\phi(r)]\\ M_{\theta\theta}&=D(1-v^{D2})[\psi(a)+\psi(r)-x^T]-B\\ &\quad(1+v^A)[\phi(a)+\phi(r)-\varepsilon^T]\\ M_{r\theta}&=0 \end{aligned} \quad (43)$$

As a special case, we consider an infinite plate in which the distributed temperature change T is given by

$$\begin{aligned} T&=T^*(x_3) \text{ for } r\le b\\ T&=0 \text{ for } r>b \end{aligned} \quad (44)$$

Taking $a\to\infty$ in (41), it can be shown that the solutions of the displacements for the infinite plate are written as

$$\begin{aligned} u_r^0&=\frac{1+v^A}{2}\varepsilon^{0*}r,\quad u_\theta^0=0\\ w&=-\frac{1+v^D}{4}x^*r^2 \text{ for } r\le b\\ u_r^0&=\frac{1+v^A}{2}\frac{b^2}{r}\varepsilon^{0*},\quad u_\theta^0=0\\ w&=-\frac{1+v^D}{4}b^2x^*\left[1+2\ln\frac{r}{b}\right] \text{ for } r>b \end{aligned} \quad (45)$$

where

$$\begin{aligned} \varepsilon^{0*}&=\frac{1}{A_{11}+A_{12}}\int_{-h^0}^{h-h^0}(C_{11}+C_{12})\,\alpha T^*(x_3)\,dx_3\\ x^*&=\frac{1}{D_{11}+D_{12}}\int_{-h^0}^{h-h^0}(C_{11}+C_{12})\,\alpha T^*(x_3)\,x_3dx_3 \end{aligned} \quad (46)$$

The displacement fields (45) are identical to those derived solving an eigenstrain problem in Beom and Earmme (1999).

In a similar manner, when the plate is clamped at its boundary $r=a$, it can be shown that the solutions of the plate are written as

$$u_r^0 = (1+v^A)\, r[\phi(r) - \phi(a)]$$
$$u_\theta^0 = 0 \tag{47}$$
$$w = -(1+v^D)\left[\int_a^r \xi\psi(\xi)\, d\xi - \frac{r^2-a^2}{2}\psi(a)\right]$$

In obtaining (47), the solutions for the corresponding uncoupled plate (Beom and Earmme, 1998) have been used. Since the displacements are determined as above, we can calculate the associated resultant stresses and moments from (2) and (40). The corresponding resultant stresses and moments are written as

$$\begin{aligned} N_{rr} &= -A(1-v^{A^2})\phi(r) - A(1+v^A)^2\phi(a) \\ &\quad + B(1+v^D)[\psi(r) - \psi(a)] \\ N_{\theta\theta} &= A(1-v^{A^2})[\phi(r) - \varepsilon^{0T}] - A(1+v^A)^2\phi \\ &\quad (a) - B(1+v^D)[\psi(r) + \psi(a) - x^T] \\ N_{r\theta} &= 0 \qquad (48) \\ M_{rr} &= -D(1-v^{D^2})\psi(r) - D(1+v^D)^2\psi(a) \\ &\quad + B(1+v^A)[\phi(r) - \phi(a)] \\ M_{\theta\theta} &= D(1-v^{D^2})[\psi(r) - x^T] - D(1+v^D)^2\psi \\ &\quad (a) - B(1+v^A)[\phi(r) + \phi(a) - \varepsilon^{0T}] \\ M_{r\theta} &= 0 \end{aligned}$$

5. Conclusions

A multilayered plate composed of thin layers of isotropic materials is analyzed. The problem for the multilayered plate with body forces is formulated by using lamination theory in which displacement fields are expressed in terms of in-plane displacements on the main plane and transverse displacement. Placing the main plane at an appropriate position in the plate, a set of equilibrium equations is shown to be written in uncoupled form, which are identical to those for an uncoupled plate such as a single layer plate. The boundary conditions, in contrast to an uncoupled plate, depend on in-plane displacement as well as on the transverse displacement. Two kinds of problems for the multilayered and graded materials with body forces, which are of particular importance in the practical application, are considered in this paper: The first one is a laminate subjected to the in-plane resultant tractions prescribed on its boundary. The second one is a laminate subjected to the specified in-plane displacements on its boundary. It is proved that the problem of in-plane stretching and transverse bending of the plate can be treated separately. Thus, the complete solutions of the problems can be obtained from the sum of solutions for an uncoupled plate, due to the linearity of the problem. As examples we consider a circular laminate clamped or simply supported on its the boundary as well as a circular laminate with a constant angular velocity. Closed forms of the solutions are obtained from the known solutions of an uncoupled plate by applying the superposition method proposed in this paper. The calculations of thermoelastic stresses and displacements in multilayered plates are also discussed. Closed form solutions are obtained for a circular laminate with distributed temperature varying in the radial direction and through the thickness.

Acknowledgment

This study was financially supported by Chonnam National University in the program, 1998.

References

Beom, H. G. and Earmme, Y. Y., 1998, "Complex Variable Method for Problems of a Laminate with Multiple Isotropic Layers," *International Journal of Fracture*, Accepted for Publication.

Beom, H. G. and Earmme, Y. Y., 1999, "The Elastic Field of an Elliptic Cylindrical Inclusion in a Laminate with Multiple Isotropic Layers," *ASME Journal of Applied Mechanics*, Vol. 66, pp. 165～171.

Finot, M and Suresh, S., 1996, "Small and Large Deformation of Thick and Thin-Film Multi-Layers: Effects of Layer Geometry, Plasticity and Compositional Gradients," *Journal of the Mechanics and Physics and Solids*, Vol. 44, pp. 683～721.

Freund, L. B., 1996, "Some Elementary Connections Between Curvature and Mismatch Strain in Compositionally Graded Thin Films," *Journal of the Mechanics and Physics and Solids*, Vol. 44, pp. 723～736.

Geiger, G., 1992, "Ceramic Coatings Enhance

Material Performance," *American Ceramic Society Bulletin*, Vol. 71, pp. 1470～1481.

Jones, R. M., 1975, *Mechanics of Composite Materials*, Scripta, Washington.

Kang, K. J. and Cho, K. Z., 1991, "An Estimation of Ballistic Limit for Ceramic-FRP Composite Armor," *KSME Journal*, Vol. 5, pp. 140～148.

Kim, Y. Y. and Yoon, M. S., 1997, "Analytic Method for the Bending Analysis of Plates," *Transactions of the Korean Society of Mechanical Engineers*, Vol. 21, pp. 697～706.

Koizumi, M., 1992, "Recent Progress of Functionally Gradient Materials in Japan," *Ceramic Engineering Science Proceedings*, Vol. 13, pp. 333～347.

Lau, J. H., 1993, *Thermal Stress and Strain in Microelectronics Packaging*, Van Nostrand Reinhold, New York.

Nakajima, K., 1992, "Calculation of Stresses in GaAs/Si Strained Heterostructures," *Journal of Crystal Growth*, Vol. 121, pp. 278～296.

Nix, W. D., 1989, "Mechanical Properties of Thin Films," *Metallurgical Transactions*, Vol. 20A, pp. 1989～2217.

Reismann, H. and Pawlik, P. S., 1980, *Elasticity: Theory and Applications*, John Wiley & Sons, New York.

Suhir, E., 1993, "Predicted Bow of Plastic Packages of Integrated Circuit (IC) Devices," *Journal of Reinforced Plastics and Composites*, Vol. 12, pp. 952～972.

Suhir, E. and Manzione, L. T., 1992, "Predicted Bow of Plastic Packages Due to The Nonuniform Through-Thickness Distribution of Temperature," *ASME Journal of Electronic Packaging*, Vol. 114, pp. 329～335.

Timoshenko, S. P. and Woinowsky-Krieger, S., 1959, *Theory of Plates and Shells*, McGraw-Hill, London.

Tummala, R. R., Rymaszewski, E. J. and Klopfenstein, A. G., 1997, *Microelectronics Packaging Handbook*, International Thomson Publishing.

Young, W. C., 1989, *Roark's Formulas for Stress and Strain*, McGraw-Hill, London.